The *a priori* in Maddy's Set Theoretic Realism

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October 5, 2002

> ROBERT: No! I'm back! I'm back in touch with the source — the font, the — whatever the source of my creativity was all those years ago. I'm in contact with it again. I'm *sitting* on it. It's a geyser and I'm shooting right up into the air on top of it.
>
> CATHERINE: My God.
>
> ROBERT: I'm not talking about divine inspiration. It's not funneling down into my head and onto the page. It'll take *work* to shape these things; I'm not saying it won't be a tremendous amount of work. It *will* be a tremendous amount of work. It's not going to be easy. But the raw material is there. It's like I've been driving in traffic and now the lanes are opening up before me and I can *accelerate*. I see whole landscapes —places for new work to go, new techniques, revolutionary possibilities.
>
> – *Proof*, David Auburn

Paul Benacerraf gives criteria for a suitable philosophical account of mathematics in his 1973 paper, "Mathematical Truth," (MT).[1] It must:

(B1) Be consistent with a semantics that assigns truth to both natural language and mathematical sentences in the same way.

(B2) Allow for the possibility of mathematical knowledge, when combined with a "reasonable" epistemology.

Benacerraf's thesis is that most accounts of mathematics "can be identified with serving one or another of these masters *at the expense of the other*".[2] However, while his two criteria have been enormously influential, the arguments defending his thesis are based on assumptions that can be disputed. Specifically, he

*As a Computer Scientist, I have done work in logic and semantics, but I have had very little formal training or experience in analytic philosophy and philosophy of mathematics. This paper was researched and executed specifically for the HPS application. It draws on material and ideas from many books and articles that I read over the vacation (all listed in the References section), as well as from informal discussions with Cristie Ellis, John Morrison, and the HPS Lent Term Epistemology Reading Group. Thanks also to Professors Peter Lipton for very helpful comments on a draft, and David Park for first getting me interested in realism.

[1]Benacerraf (1983a)

[2]Benacerraf (1983a, pg. 403)

assumes a correspondence theory of truth á la Tarski in showing that "combinatorial" accounts[3] fail to satisfy (B1) and a causal theory of knowing to show that realist/Platonist accounts can't satisfy (B2). I will attempt to strengthen Benacerraf's argument by showing that it can be modified to work even if these assumptions are rejected. I will also consider an account of mathematical objects given by Benacerraf's student, Penelope Maddy, in (Maddy, 1990). Like Robert in my epigraph, Maddy claims to be able to see mathematical objects (though she means it literally, while he is speaking in metaphor). While Maddy's brand of realism, based on perception, satisfies both of Benacerraf's criteria, by explaining sets as real entities in the physical universe, her theory describes mathematics as an empirical rather than *a priori* science.

Thus, Maddy's set theoretic realism gives us mathematical truth explained in a way consistent with natural language semantics and allows us to have knowledge of these truths, but at the price of placing mathematics on the same level of certainty as the empirical sciences.

The Working Mathematician's Account

Many working mathematicians think of mathematical objects in real terms. Robert's fictional character in the passage from the play *Proof* above, is typical. An academic who has been unstable and unable to work for several years, he has a sudden, unexplained remission of his symptoms and finds himself able to do mathematics again. Robert describes the mental state which allows him work on mathematics again in very physical terms: "[being] in touch with the source", "shooting right up into the air", and "see[ing] whole landscapes".[4] This general view — that mathematical objects are real entities not fictions, and that mathematical truths are discovered not synthesised — is popular among mathematicians. Kurt Gödel argued for an extreme form of realism known as Platonism where mathematical objects exist in a reality that is separate from the physical universe:

> the objects of transfinite set theory... clearly do not belong to the physical world and even their indirect connection with physical experience is very loose.[5]

The mathematician Alain Connes expresses similar beliefs: "When I speak of the independent existence of mathematical reality, I expressly do *not* locate it in physical reality".[6]

Platonism is attractive ontologically because it captures the familiar, traditional notion that mathematical truths are objective truths. That is, they are not mere fictions created by the mind. For example, most people would agree that '5 + 7 = 12' is true independent of any human formulating a theory of simple arithmetic (by either creative intuition or generalising from an empirical observation). Though they don't dispute that our understanding of

[3]Benacerraf defines combinatorial accounts as those that define mathematical truth "on the basis of certain (usually proof-theoretic) syntactic facts about them".(Benacerraf, 1983a, pg. 407) Thus, Benacerraf lumps formalism, conventionalism, and logicism (and possibly others) together as combinatorial.

[4]Auburn (2001, pg. 70)

[5]Gödel (1983, pg. 484)

[6]Changeux and Connes (1995, pg. 28)

mathematics is influenced by a complex cultural and intellectual evolutionary history, they claim that the truths would still hold even if we never developed an understanding of mathematics. Moreover, the rules of arithmetic is necessary: given a realist ontology and the regular meanings of the symbols '5', '7', '12', '+', and '=', we couldn't have $5+7$ equal anything else but 12. Thus, Platonism gives a very favourable account of mathematics, ontologically.[7] From an epistemological viewpoint however, Platonism has deep problems. If mathematical objects exist in a reality that is separate from our universe and thus, all of our normal sensory faculties, it is hard to see how we can gain any knowledge of that entities in that universe. A typical answer is that some sort of mathematical intuition, distinct from perception, allows the mathematician to interact with mathematical reality. Connes writes:

> The mathematician develops a special sense, I think — irreducible to sight, hearing, or touch—that allows him to perceive a reality every bit as constraining as physical reality[.][8]

Gödel also posits "mathematical intuition", analogous to perception as our source of knowledge about medium-sized objects in physical reality, as the source of our knowledge of mathematical objects.[9]

Mathematicians who are otherwise realists but do not wish to defend an extra-terrestrial Gödelian intuition when pressed on epistemological issues, often turn to formalism. Formalists hold that mathematics is not about anything; it is merely the study of deduction, understood syntactically. Mathematical truths for the formalist are without meaning: they are merely the result of mechanical symbol manipulation according to axioms and rules. These mathematicians practice philosophical "double-think":[10]

> the typical working mathematician is a [realist] on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality whose properties he is attempting to determine. But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all.[11]

However, while formalism provides an alternative that avoids the epistemological problems associated with Platonism, it is unsettling in that it disconnects the meaning of mathematics from reality. This is troubling given the effectiveness of mathematics in describing the physical world. As Maddy puts it,

> It isn't hard to see how various true statements of mathematics can help me determine how many bricks it will take to cover the back patio, but how can a meaningless string of symbols be any more relevant to the solution of real world problems than an arbitrary arrangement of chess pieces?[12]

[7] Steiner (1973, pg. 57)
[8] Changeux and Connes (1995, pg. 28)
[9] Gödel (1983, pg. 484)
[10] Maddy (1990, pg. 3)
[11] Davis and Hersh, as quoted in Maddy (1990, pg. 3)
[12] Maddy (1990, pg. 24)

It is precisely these issues — the epistemological deficiencies of Platonism, and the unsatisfying metaphysics that results from a formalist account of mathematics — that Benacerraf grapples with in MT.

(B1): Characterising Mathematical Truth

Benacerraf's claim that combinatorial accounts of mathematics fail to satisfy (B1) assumes a Tarskian correspondence theory of truth. Correspondence theories of truth hold that the truth of sentences or propositions are solely determined by factual content in reality.[13] Alfred Tarski tells us how to understand this correspondence terms of semiotics. Benacerraf's simplification of Tarski is as follows: "its essential feature is to define truth in terms of reference (or satisfaction) on the basis of a particular kind of syntactico-semantical analysis of the language".[14] In other words, to determine the truth of a sentence, we examine the referents in the sentence and determine if they are in the relation that the sentence describes. For example, "snow is white" is true if and only if photons bounce off of the stuff that we call snow at the wavelength that we call white (or in simpler terms, if real snow is the real colour white). Thus, Tarski reduces truth to other semantic concepts (according to Hartry Field in his 1972 paper "Tarski's Theory of Truth", these are, "what it is for a name to denote something, what it is for a predicate to apply to something, and what it is for a function to be fulfilled by some pair of things"[15]).

Benacerraf claims that if we accept Tarski's referential theory of truth for mathematical sentences for "snow is white," then in order to remain consistent, we must also use the theory for sentences like the following:

(m) there exist at least two prime numbers between 15 and 20

It is unacceptable to define mathematical truth as derivability in a formal system because it fails to give us a consistent over-all theory when combined with a normal semantics for natural language sentences. The claim is that we cannot say that the truth of "snow is white" is obtained by doing a semantical analysis along referential lines but that (m) should be analysed only purely syntactically (*i.e.*, the ability of a symbol manipulating machine to produce it from a set of base axioms). If we want to say that (m) is true, then we are forced to say that '15' and '20' are names and the following predicate P applies the referents of '15' and '20'; *i.e.*, $\mathsf{P}(15, 20)$:

$$\mathsf{Prime}(x) \stackrel{\text{def}}{=} (2 \leq x) \; \& \; (\neg(\exists y(\exists z(2 \geq y \; \& \; 2 \geq z \; \& \; y * z = y))))$$
$$\mathsf{P}(x, y) \stackrel{\text{def}}{=} \exists n(\exists m(x \leq n \leq y \; \& \; x \leq m \leq y \; \& \; \mathsf{Prime}(m) \; \&\mathsf{Prime}(n)))$$

Benacerraf assumes a correspondence theory of truth in showing that combinatorial theories fail to satisfy (B1). However, not everyone accepts that a correspondence theory of truth is required for a reasonable ontology. Instead perhaps, the completely degenerative (in the strength of the references that it requires) disquotational account of truth, that "snow is white" is true is true just if snow is white (without following the references to real objects as above) is enough. On this basis, Maddy argues that Benacerraf cannot claim that a

[13]Blackburn and Simmons (1999, pp. 1-2)
[14]Benacerraf (1983a, pg. 408)
[15]Field (1972, pg. 350)

"robust theory of the referential connections"[16] between numbers as names and numbers as entities is necessary. Thus the disquotational theory would relieve some combinatorial accounts from Benacerraf's critique. If Tarski's referential semantical theory is not needed for other areas of discourse, then the argument that mathematical truths must be handled consistently and in accordance with a correspondence theory of truth is not supported.

Even if one accepts the assumption of a correspondence theory, I claim that Benacerraf's argument that mathematical truth must be about real objects is circular. First, as described above, Field shows that Tarski defines truth in terms of other semantic concepts including predicate satisfaction and function application. It would be hard to describe these concepts without making reference to mathematical entities: relations, sets, etc. Thus, it is fruitless to try to reason about the nature of mathematical truth when mathematics is needed to define our theory of truth.

The second bit of question begging is in arguing for realism in mathematics while assuming a correspondence theory of truth. Benacerraf claims that the combinatorial account cannot describe truth in a manner which is consistent with Tarskian correspondence theory. Modulo the assumption of such a theory, he wants to force the combinatorialist into admitting that a mathematical sentence is true is to claim that it is about the world. What Benacerraf has not shown is that *any* reasonable theory of truth cannot be combined with a combinatorial account of mathematical truth consistently; he has only handled one particular case. By assuming a correspondence theory, his conclusion is supported. However, he does not show that a correspondence theory is the best choice; still a difficult and non-trivial task.

The assumption that truth must correspond to reality is precisely what the formalist denies. The formalist calls into question the claim that truth could just claim that when she says that (m) is true, she does not mean to imply a truth about reality in the Tarskian, correspondence sense. Rather, truth about mathematical propositions should be taken as a much weaker concept; a mere abbreviation for the ability to be derived in a formal system from a set of axioms and rules. So when we say "there exist at least two..." is true, we really mean that "there exist at least two..." is derivable from the axioms and rules of a suitably powerful formal system. She could continue, truth as defined by a theory satisfying Tarski's correspondence theory and truth as derivability of a mathematical proposition from a formal system are two unrelated concepts – the fact that we use the same name for them is merely a result of projecting our notion of truth onto the formalist's mathematical propositions. In particular, if she rejects the correspondence theory for a disquotational account of truth, then the inconsistencies that Benacerraf highlights disappear.

Benacerraf's conclusion — that combinatorial accounts of mathematics give an unsatisfactory account of truth — can be revived if it is coupled with an indispensability argument. Maddy describes such an argument:

> On the naturalized approach, we judge what entities there are by seeing what entities we need to produce the most effective theory of the world.[17]

[16] Maddy (1996, pg. 64)
[17] Maddy (1990, pg. 29)

As mathematics is a critical part of our best theory of the world[18] we insist that mathematical truths are about reality, not merely in that they are derivable in a formal system. For example, we cannot deny that the concepts of truth that result from deriving $4 \times 4 = 16$ from Peano's axioms and that a covering 4m $\times$ 4m room requires 16m^2 of tile are unrelated. To maintain that truths derived from Peano axioms are not about the world is to deny the scientific results, based on mathematics that help us understand physical reality.

Thus a philosopher exploring Benacerraf's requirement for mathematical truth can still reach the same conclusion: that combinatorial accounts of mathematics fail to explain why mathematics is effective in describing reality. However, the argument rests on an indispensability claim, not a purely semantic notion of truth as Benacerraf attempted to show.

(B2): Obtaining Mathematical Knowledge

Benacerraf's critique of Platonist accounts should, like (B1), be understood in the historical context in which it was written. The traditional account of knowledge, from Plato, says that X knows p if X has justified true belief that p. However, Gettier showed that this account fails with a counter-example (Alvin Goldman's formulation):

> Smith believes
>
> (q) Jones owns a Ford
>
> and has very strong evidence for it ... Smith has another friend, Brown, of whose whereabouts he is totally ignorant. Choosing a town quite at random, however, Smith constructs the proposition
>
> (p) Either Jones owns a Ford or Brown is in Barcelona.
>
> Seeing that q entails p, Smith infers that p is true. Since he has adequate evidence for q, he also has adequate evidence for p.[19]

However, suppose that Smith's evidence for q is mistaken, while p happens to be true because Brown by chance actually is in Barcelona. Then we cannot say that Smith knows p even though p is true, and Smith has justified belief for p — the fact that p turns out to be true is purely an accident. As a solution to this style of problem, Goldman suggests a causal requirement for knowledge:

> One thing that seems to be missing in this example is a causal connection between the fact that makes p true ... and Smith's belief of p. The requirement of such a *causal connection* is what I wish to add to the traditional analysis.[20]

It is in this epistemological climate that Benacerraf wrote MT. Presupposing a causal theory of knowledge, it is easy to show the epistemological problems

[18] Field's attempt to rewrite science without mathematics (Field, 1980), and Maddy's own doubts (Maddy, 1996, pp. 65-66) may undercut this Quinean argument. I do not wish to comment on that debate yet, but merely to draw out an alternate objection to combinatorial accounts that doesn't depend on a correspondence theory of truth.

[19] Goldman (1967, pg. 357)

[20] Goldman (1967, pg. 358)

with Platonism. If mathematical objects are independent of our spatio-temporal reality, as for the Gödelian Platonist, then we cannot have any mathematical knowledge because we lack any causal connection to mathematical objects. If X is a human and p an arbitrary mathematical truth that X purports to know, Benacerraf argues that with a Platonist account:

> We think that X could not know that p. What reasons can we offer in support of our view? If we are satisfied that X has normal inferential powers, that p is indeed true, etc., we are often thrown back on arguing that X could not have come into possession of the relevant evidence or reasons: that X's four-dimensional space-time worm does not make the necessary (causal) contact with the grounds of the truth of the proposition for X to be in possession of evidence adequate to support the inference.[21]

The Platonist account then, fails to show how an individual can have knowledge of any mathematical truths because of the lack of any causal connection between the abstract mathematical objects that the truths are about and the individual. However, we clearly do have knowledge of some mathematical truths. Thus, Platonism is an epistemologically flawed position.

Causal theories of knowing have not survived the intervening 30 years since Goldman's original formulation unscathed. Goldman himself gives a counter-example that shows that a causal theory of knowledge also has problems. His fake-barn example goes like this. Suppose Henry drives through the countryside, and sees a barn next to the road. He is in possession of what we'd normally grant is adequate justification for him believing that the object he sees is a barn:

> each object is fully in view, Henry has excellent eyesight, and he has enough time to look at them reasonably carefully, since there is little traffic to distract him.[22]

Further, the object that he perceives gives the necessary causal connection between his perceptions and beliefs that Goldman proposed in (Goldman, 1967). Thus, we say that Henry has knowledge that the object in his view is a barn. However, what if we find out that, "unknown to Henry, the district he has just entered is full of papier-mâché facsimiles of barns",[23] that are indistinguishable from real barns? Goldman claims that with this further information we would want to deny the claim that Henry knows that the object he sees is a barn because he wouldn't be able to discriminate a real barn from a facsimile. However, neither the traditional nor causal accounts provide any criteria for distinguishing the case where we don't know about the fake barns (and presumably would assign knowledge) from the case where we are told that the district is full of them (and we would deny him knowledge). The justified-true-belief account doesn't help us — Henry satisfies all three conditions in both cases; nor does the causal account — the object in his view, facsimile or not, provides the necessary causal connection. Thus Goldman "abandon[s] the requirement that a knower's belief that p be causally connected with the fact, or state of affairs, that p"[24]

[21] Benacerraf (1983a, pg. 413)
[22] Goldman (1976, pg. 772)
[23] Goldman (1976, pg. 773)
[24] Goldman (1976, pg. 771)

To replace the discarded causal theory, Goldman suggests reliablism. Rather than simply insisting on a causal connection between p and the individuals belief that p, he suggests that the connection must be reliable in that it "would produce true beliefs, or at least inhibit false beliefs, in relevant counterfactual situations."[25] Expanding on this notion, he writes:

> A person knows that p, I suggest, only if the actual state of affairs in which p is true is *distinguishable* or *discriminable* from him by a relevant possibly state of affairs in which p is false.[26]

With this reliablist account of knowledge, Goldman is able to uniformly deny Henry's knowledge that the object he sees is a barn because with the possibility of fake barns being a relevant consideration in the area where he is driving, Henry's perceptions would not be able to distinguish between a real barn and a facsimile.

If the causal account of mathematics is largely ignored in the contemporary literature, what remains of Benacerraf's claim that Platonist accounts fail to be epistemologically satisfactory? As a theory, Platonism still fails to provide any positive account for mathematical knowledge. As Mark Steiner writes: "clearing the site is not building the house".[27] Or as Maddy puts it, "the Platonist still owes us an explanation of how and why [set theorist] Solovay's beliefs about sets are reliable indicators of the truth about sets".[28] Goldman's reformulated account of knowledge given in (Goldman, 1976) gives reliability as a criterion. The burden remains on the Platonist to show a reliable connection (causal or otherwise) between the separate world of mathematical entities and a human being. Intuition itself is hardly reliable as shown by the many historical errors in mathematics that were propagated because of their supposed intuitive obviousness;[29] thus the possibility of Gödel's intuition providing an epistemological connection for mathematics is further damaged under a reliablist account. Field writes: "[Benacerraf's challenge is] a challenge to our ability to *explain the reliability* of our mathematical beliefs".[30]

While causal theories of knowledge have become less popular among since MT was written, the spirit of Benacerraf's challenge to Platonists remains forceful. Platonists still offer no concrete account of how we could come to have any knowledge of mathematical truths, in a reasonable epistemological package.

Maddy's Set Theoretic Realism

Benacerraf presents a dilemma in MT: traditional realist accounts of mathematics that give us meaningful mathematical truth do not explain how we can acquire mathematical knowledge. However, his analysis leaves out other styles of realism that might be able to satisfy his requirement for truth while avoiding the epistemological difficulties of Platonism. Maddy offers a realist account based on perception that meets both of Benacerraf's requirements.

According to Maddy, we literally perceive sets through our optical senses, much in the same way that we see medium-sized objects (assuming a realist

[25] Goldman (1976, 771)
[26] Goldman (1976, pg. 774)
[27] Steiner (1973, pg. 66)
[28] Maddy (1990, pg. 43)
[29] John Morrison pointed this out to me.
[30] Field, as quoted in Maddy (1996, pg. 62)

ontology). As the brain develops, it sets aside certain areas dedicated to recognising sets. Thus, we learn the ability to recognise and identify sets as real entities in the physical world. Her account is as follows:

> My claim [is that] we can and do perceive sets, and that our ability to do so develops in much the same way as our ability to see physical objects. Consider the following case: Steve needs two eggs for a certain recipe. He opens the carton and sees, to his relief, three eggs there. My claim is that Steve has perceived a set of three eggs... this requires that there be a set of three eggs in the carton, that Steve acquire perceptual beliefs about it, and that the set of eggs participate in the generation of these perceptual beliefs in the same way that my hand participates in the generation of my belief that there is a hand before me when I look at it in good light.[31]

Thus, knowledge of mathematical truths about sets are generated by physical processes similar to the causal processes that give us knowledge of medium-sized objects. In this way, Maddy makes her account of set perception as strong as the case for object perception (which, presumably, few would deny).

Maddy defends her claim against two critics: first, the nominalist who says sets need not exist and second, the Platonists who says that they should not be located in physical reality. To the first objection, Maddy cites an indispensability argument. Sets are foundational to much of mathematics, which is in turn foundational to our "best theory of the world."[32] Therefore, if there is such a thing as reality, sets must be included in it. To the Platonist who claims that mathematics must be independent of space and time, she replies that there is:

> no real obstacle to the position that the set of eggs comes into and goes out of existence when they do, and that, spatially as well as temporally, it is located exactly where they are.[33]

To summarise Maddy's position, it is that sets exist, as entities in the physical universe. We are able to perceive them directly, just as we perceive objects in the physical world, and they are located in space and time with their elements.

What Could Numbers Be?

Before examining whether Maddy's account of set perception satisfies the criteria in MT it will be worthwhile to examine how a more complete account of mathematics can be expanded from this simpler theory. Maddy does not claim that all of mathematics can be given a foundation on intuitive objects, such as sets — the failure of formalism revealed by Gödel would make us doubt any such claim. However, as Russell and Whitehead showed (Russell and Whitehead, 1913), much of mathematics can be *reduced* to a (perhaps non-intuitive) logic. Thus, by giving an account of sets, portions of mathematics can be explained in terms of logic and basic set theory. In particular, according to Maddy the natural numbers are just properties of sets.

Benacerraf's famous ontological challenge to Platonism (Benacerraf, 1983b) claims that numbers could not be sets. He shows that the numbers, and basic

[31] Maddy (1990, pg. 58)
[32] Maddy (1990, pg. 59)
[33] Maddy (1990, pg. 59)

truths of arithmetic can be derived in multiple ways from basic set theory and logic. He then argues that argues that if numbers are sets, that the numbers must *really be* one particular ω-sequence of sets. However, we have no reason to pick any adequate collection of sets over another:

> Our present problem is to see if there is one account [of numbers as sets] which can be established to the exclusion of all others, thereby settling the issue of which sets the numbers really are. And it should be clear by now that there is not. Any purpose we may have in giving an account of the notion of number ... will be equally well (or badly) served by any one of the infinitely many accounts[.][34]

Therefore, according to Benacerraf, numbers could not be sets (or indeed, any objects, though I will not pursue that argument here).

Maddy's concedes that numbers are not sets. However, she would like to avoid "adding a new type of entity to her ontology."[35] Thus, she reduces the idea of number to a property of sets. Rather than understanding the numbers as some particular collection of sets, ordered by a less-than relation, we should understand the numbers as the property of sets. Just as length itself is not defined by a particular ruler or scale of measurement but rather as a property of spatial relations between medium-sized objects, so number is not a particular entity, but a property determined by one-to-one correspondences between sets.[36] She explains number in primitive set theoretical terms as follows. Two sets have the same number property if they can be put into a one-to-one correspondence. A set has a greater number property than another set if "there is a one-to-one correspondence between [its] members ... and a proper subset of [the other set]."[37] Expanding this notion fully, it is easy to see that the numbers can be explained as a property of sets.

Seeing how Maddy constructs her two-tiered approach to a philosophy of mathematics — sets directly perceived, more complex mathematical entities reduced to logical consequences or properties of sets — we can turn to evaluating how her account stacks up against Benacerraf's criteria given in MT. It seems clear that (B1) is easily satisfied. Whatever semantic theory for defining truth is chosen (correspondence, disquotational or otherwise), it will treat mathematical entities in the same way that it treats objects in physical reality because mathematical entities *are* physical entities according to Maddy. Thus, a mathematical sentence is true or false in that it is about reality (for a correspondence theory) just as for natural language sentences. We determine the truth or falsity of "snow is white", "$5+7=12$" and "$\aleph_1 = c$"[38] in the same way: by examining the state of things in the world.

However, unlike Platonist ontologies, Maddy's set theoretic realism does not make knowledge of mathematical truths impossible. By placing sets in the physical universe, mathematical knowledge is generated by the same kinds of reliable processes that cause me to believe truths about medium-sized objects. Thus, (B2) is also satisfied by Maddy's set theoretic realism.

[34] Benacerraf (1983b, pg. 284)
[35] Maddy (1990, pg. 87)
[36] Maddy (1990, pg. 88)
[37] Maddy (1990, pg. 88)
[38] One formulation of Cantor's Continuum Hypothesis.

Chihara's Objection

Maddy's simple account of sets as real objects in the physical world is convincing, and seems to satisfy Benacerraf's criteria for a suitable account of mathematical truth and knowledge. However, a serious problem was noticed by Chihara. He imagines a single apple sitting on his desk. According to Maddy, when he sees the apple, he also sees the unit set containing the apple. However, Chihara notes that from perceptions alone there is no way to distinguish the apple itself from the unit set containing the apple. He writes:

> One wonders how this object is to be distinguished (perceptually) from the apple, since it has exactly the same shape and colour. Perhaps it feels different. Let's touch it. But I can't feel anything different from the apple. How about its smell or taste? Again, it would seem that the set must be identical in smell and taste to the apple. So it looks, feels, smells, and tastes exactly like the apple and is located in exactly the same spot and at exactly the same time — yet it is a distinct entity![39]

He goes on to note that just as we have no way of discriminating between an object and its unit set, we have similar problems finding any way to distinguish between the sets containing the set containing the apple and the apple (*i.e.* $\{\mathsf{apple}, \{\mathsf{apple}\}\}$) and so on.

Maddy offers two solutions to this problem. First, the set theoretic realist can insist that though there is a difference between the apple and the unit set, it isn't a perceivable difference. However, this would cause serious epistemological problems for a reliablist because unperceivable yet significant differences in propositions (like facsimile barns), when relevant, rule out knowledge of those propositions. The second option that she presents is simply to insist that there is no difference between a physical object and its unit set. Thus, $\mathsf{apple} = \{\mathsf{apple}\}$. However, for higher-order sets, the unit set containing a set is not the same as the set itself. This requires a very simple kind of type theory to distinguish between sets and their elements and avoid collapsing many unit sets down to their elements. The axiom for a set theoretic realist that $x = \{x\}$ is only true if x is not itself a set. Maddy makes this distinction to preserve the usual properties of sets (such as number):

> This is not to say that every singleton is identical with its sole member; there is every reason to distinguish between $\{\{0, 1, 2, 3...\}\}$ and $\{0, 1, 2, 3...\}$, starting with the fact that one is finite and the other infinite.[40]

This simple modification of the axioms takes care of Chihara's direct objection, but a deeper problem remains. The crux of Chihara's objection is that the common mathematical notion of sets as defined by axioms does not necessarily match up with sets perceived as Maddy describes. Thus, the set $\{\mathsf{apple}, \{\mathsf{apple}\}\}$, which is a perfectly reasonable set if generated from the axioms, causes problems when a set theoretic realist tries to explain how it can be perceived. There are many more sets that are valid from a set theoretical point of view, but not necessarily perceivable.

[39] Chihara, as quoted in Maddy (1990, pg. 151)
[40] Maddy (1990, pg. 153)

A challenge similar to Chihara's goes like this. It is unclear how a human could perceive the difference between the set $\{\mathsf{apple}, \{\emptyset\}\}$ and the unit set $\{\mathsf{apple}\}$. There are important set theoretical differences: the first can be put into a one-to-one correspondence with the von Neumann ordinal for 1 ($\{\emptyset\}$) while the second has the same cardinality as the von Neumann ordinal for 2 ($\{\emptyset, \{\emptyset\}\}$). However, Chihara's argument can be extended to this case too: how can we distinguish perceptually the set with cardinality 2 from the singleton set? It may be possible to fix this problem by adding another axiom or special case.[41] However, Chihara's fundamental concern remains: sets as described by axioms do not necessarily match up with the explanation of sets as perceptual objects. Mathematicians already struggle to find good axioms for set theory. Adding the requirement that the axioms match up with the epistemological issues surrounding a particular account of mathematics based on perception makes the task even harder. Will we require that the axioms of set theory be consistent with Maddy's account or with modern mathematics? Is it possible to find axioms such that the two are the same? While I think that the Maddy's account of sets as real perceived objects is on the right track towards a reasonable realist account, it must be careful to avoid redefining mathematics in order to make the philosophical account sound.

Empirical Mathematics

Benacerraf's criteria given in MT captures an important distinction between traditional accounts of mathematics. The distinction is so profound, that it can be easily rescued from their historical arguments with minor modifications. Modernised, his argument goes like this. Realist accounts are satisfying ontologically, in that they extend the common notion of truth to mathematics, but are epistemologically deficient. Combinatorial accounts explain truth as a syntactic rather than semantic feature of mathematical languages; thus, they offer easy access to mathematical knowledge. Unfortunately, they disconnect mathematical truth from reality which is unsatisfying given the necessity of mathematics to scientific theories.

Maddy's set theoretic realism satisfies both (B1) and (B2) by placing sets, and thus, mathematical truth, in physical reality. Mathematical truths for her are just truths about the world. Moreover, we can discover these truths through our senses, just as we discover ordinary non-mathematical facts. This simple account, which has the advantages of realism but avoids the problems with Platonism, puts mathematical knowledge on the same level as our empirical knowledge about the world.

The empirical nature of mathematical objects described by Maddy's set theoretical realism is a significant departure from traditional accounts. Platonists of course, hold that mathematical truths are knowable *a priori* because they are about entities in a reality that is separate from our perceptual facilities. By definition if they are knowable (which of course is the big problem for Platonists) then they are knowable without special experience; *i.e.* they are knowable *a priori*. The Platonist's mathematical intuition is analogous to the intuition of

[41]Perhaps Maddy's existing rule that $x = \{x\}$ does it; however, it's not clear how this should apply to the null set – does $\emptyset = \cdot$? What is the physical object contained in the null set?

rationalists; they explain our knowledge of *a priori* truths as being recognised "by the light of understanding".[42]

Traditional empiricism denies that knowledge can be gained from anything but experience, and that empirically validated truths are ever certain. Ayer describes this view, due to Hume:

> No general proposition whose validity is subject to the test of actual experience can ever be logically certain. No matter how often it is verified in practice, there still remains the possibility that it will be confuted on some future occasion. The fact that a law has been substantiated in $n - 1$ cases affords no logical guarantee that it will be substantiated in the nth case also, no matter how large we take n to be.[43]

However, this view, which we accept for scientific theories such as physics, chemistry, and biology is not usually extended to mathematics. Ayer writes "whereas a scientific generalisation is readily admitted to be fallible, the truths of mathematics appear to everyone to be necessary and certain".[44]

Ayer's way out of this difficulty, is to explain mathematical truths, and indeed all *a priori* truths as analytic, in the Kantian sense. They are true precisely because of their meaning and not because of any correspondence to factual reality. He writes:

> The principles of logic and mathematics are true universally simply because we never allow them to be anything else. ...the truths of logic and mathematics are analytic propositions or tautologies.[45]

Thus, for Ayer, mathematical truths are without factual content. They are true, but not about the world. That an *a priori* truth is true about the world cannot be known itself *a priori*. For example, the geometry seems an accurate way of describing the 3-D spatial reality that we live in. However, for Ayer, Euclid's geometry is not about the world. It is true because it is constructed to be so. Any connection to the world is an empirical conclusion, and thus fallible. For Ayer, "Whether a geometry can be applied to the actual physical world or not, is an empirical question which falls outside of the scope of the geometry itself".[46]

Maddy's account of mathematics is a departure from both of these explanations of *a priori* truths. Her theory was formulated precisely to avoid the mystical spookiness of Platonism and rationalism. Thus, she would undermine all of her efforts to anchor mathematics in reality by appealing to an intuitive faculty to explain the *a priority* of mathematics. However, because her account is based on an indispensability argument – that mathematics is fundamental to our best theory of the world, she cannot take Ayer's position that mathematical truths are purely analytic and not about the world. Maddy quietly admits this fact,[47] but is not overly concerned that her account removes the certainty from mathematics.

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[42] Wright as quoted in Cassam (2000, pg. 48)
[43] Ayer (1983, pg. 315)
[44] Ayer (1983, pg. 316)
[45] Ayer (1983, pg. 319)
[46] Ayer (1983, pg. 324)
[47] Maddy (1990, pg. 155)

A way out of this problem for the set theoretic realist is to dispute the fact that mathematical truths are necessary, certain, and knowable *a priori*. The historical development of mathematics as a discipline certainly contains errors and revisions of what was thought to be certain mathematics.[48] Mathematics is more like the natural sciences than most accounts care to admit.

Maddy's account may be the only way to reconcile the desire for realism motivated by concerns about truth, with a reasonable epistemology as Benacerraf suggests in MT. However, making mathematics about physical reality takes away the certainty that most other accounts normally associate with mathematical truths. Perhaps what we really want is an account that satisfies (B1) and (B2) as well as

(B3): mathematical truths are certain, necessary, and knowable *a priori*

However, the combination of these three criteria is, I suspect, enough to rule out all reasonable accounts of mathematics. In the spirit of naturalism that Maddy advocates, we have to settle for a realist account of mathematics that is only as certain as our best scientific theories.

[48]Hersh (1997)

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